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Discussion

From imprecise to granular probabilities

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Abstract

Gert de Cooman's work is an important contribution to a better understanding of how to deal with imprecise probabilities. But there is an important issue which is not addressed. How can imprecise probabilites be dealt with not in isolation but in the broader context of imprecise probability distributions, imprecise events and imprecise relations? What is needed for this purpose is the concept of granular probability—a probability which is defined in terms of a generalized constraint of the form X isr R, where X is the constrained variable, R is a constraining realtion of r is an indexing variable which defines the modality of the constraint, that is, its semantics. A few examples are used as illustrations.

Keywords: Imprecise probabilities; Granular probabilities; Generalized constraint; Precisiation

Dr. de Cooman's paper is clearly an important contribution to a better understanding of how to deal with imprecise probabilities—an issue which has rapidly grown in visibility and importance during the past decade—and especially since the publication of Peter Walley's classic "Statistical Reasoning with Imprecise Probabilities," [1]. But there is a basic issue which is not fully addressed in de Cooman's treatise. This issue is related to the views expressed in my JSPI paper [3].

To make my point, I will use three relatively simple examples. (a) A box contains balls of various sizes and weights. Most are large; and a large ball is likely to be heavy. What is the probability, p, that a ball

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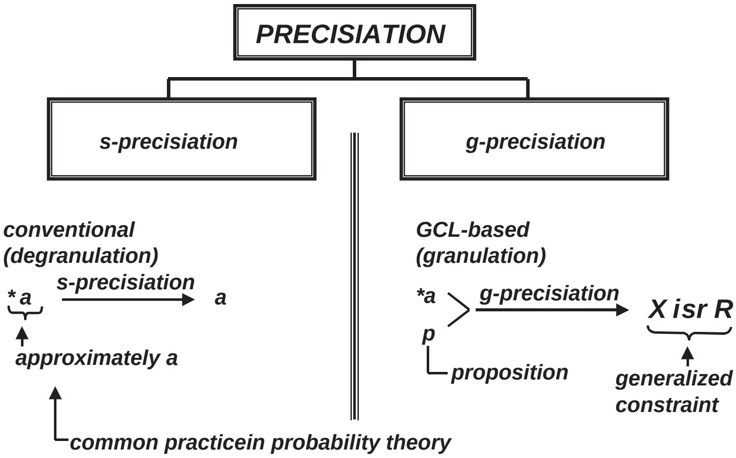

Fig. 1. Modes of precisiation.

drawn at random is not heavy? (b) X is a real-valued random variable. Usually, X is small, where small is a fuzzy subset of the real line. What this means is that the fuzzy probability of the fuzzy event "X is small" is usually, where usually plays the role of a fuzzy number. Let f be an imprecisely defined function from reals to reals which takes X into Y. For concreteness, assume that f is defined through a collection of fuzzy if–then rules, e.g., if X is small then usually Y is small; if X is medium then usually Y is large; and if X is large then usually Y is small. Let q be the probability of the fuzzy event "Y is not large." What is q? (c) X is a real-valued random variable; a and b are real numbers, with $a < b$. What we know about X is that: (1) Prob (X is much longer than approximately a) is high; and (2) Prob (X is much smaller than approximately b) is high. What is the probability, r, of the event "X is approximately $(a + b)/2$"? What is the expected value of X? How would de Cooman's techniques apply to these examples?

Clearly, p, q and r are imprecise probabilities. But the problem is that there is no reason to assume that they are representable as imprecise probabilities as defined in de Cooman's work, e.g., as intervals, fuzzy sets, possibility distributions or second-order possibility distributions. What is needed to represent p, q and r is the concept of granular probability—a concept which is more general than that of imprecise probability in de Cooman's paper.

More concretely, let me begin by raising a basic question: How can we define precisely what is meant by "imprecise probability." A broader question is: How can we precisiate the meaning of "approximately a," denoted by $^*a$, where a is a real number? With reference to Figs. 1 and 2, there is a hierarchy of ways in which this can be done, with the simplest and most commonly used way, call it s-precisiation (s standing for singlular), amounting to ignoring that $^*a$ is imprecise and treating $^*a$ as a. This is what is commonly done in many probabilistic computations—manipulating imprecise subjective probabilities as if they were precise.

In g-precisiation, with g standing for granular, $^*a$ is treated as a granule, with a granule being a clump of values drawn together by indistinguishability, equivalence, similarity or proximity. The simplest form of a granule is an interval. Other forms are shown in Fig. 2. The most general element of the hierarchy is what is referred to as a generalized constraint (Fig. 1). A generalized constraint is an expression of the form

$$X \text{ isr } R,$$

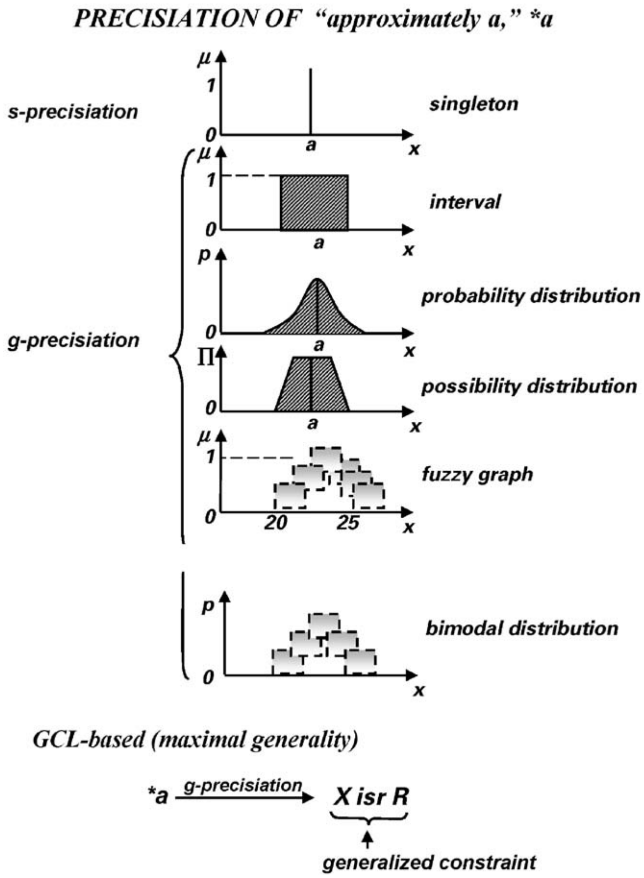

Fig. 2. Hierarchy of precisiations of $^*a$.

where *X* is the constrained variable, *R* is a constraining non-bivalent (fuzzy) relation, and *r* is an indexing variable which identifies the modality of the constraint. The principal modalities are: possibilistic (r = blank); veristic ($r = v$); probabilistic ($r = p$); usuality ($r = u$); random-set ($r = rs$); and fuzzy-graph ($r = fg$). A composite generalized constraint is derived from other generalized constraints through combination, qualification and constraint propagation. The set of all composite generalized constraints, together with the rules governing generalized constraint propagation, constitutes what is called the Generalized Constraint Language (GCL). The GCL is the centerpiece of precisiated natural language (PNL) [4]. In PNL, the meaning of a concept or a proposition is precisiated through translation into GCL.

To return to our discussion of imprecise and granular probabilities, a granular probability will be understood to be an imprecise probability whose meaning is precisiated through translation into the Generalized Constraint Language. In this sense, the imprecise probabilities $p,\ q$ and *r* in Examples (a), (b) and (c) are granular probabilities. Computation of granular probabilities requires application of the rules which govern propagation of generalized constraints in GCL. As an illustration, let us consider Example (c). Assume that $^*a,\ ^*b,\ ^*(a+b)/2$ and "high" are defined as fuzzy intervals, and "much larger" and "much smaller" are defined as fuzzy relations. Let *g* be the probability density of *X*. Using the definition of the probability measure of a fuzzy event, the generalized constraints induced by Conditions

(1) and (2) may be expressed as

$$\text{Condition (1)}: \int_R \mu_{(^*a \circ \text{much larger})}(u) g(u)\, \mathrm{d}u \text{ is high;}$$

$$\text{Condition(2)}: \int_R \mu_{(^*a \circ \text{much smaller})}(u) g(u)\, \mathrm{d}u \text{ is high;}$$

where R is the real line; $\circ$ denotes composition; and μ is the membership function of its subscript. Forming the conjunction of Conditions (1) and (2), and assuming that $\mu_{\text{high}}(u)$ is monotone nondecreasing, the generalized constraint on g may be expressed as

$$\mu(g) = \mu_{\text{high}}\left(\left(\int_R \mu_{(^*a \circ \text{much larger})}(u) g(u)\, \mathrm{d}u\right) \wedge \left(\int_K \mu_{(^*b \circ \text{much smaller})}(u) g(u)\, \mathrm{d}u\right)\right),$$

where $\wedge$ is min or, more generally, a t-norm, and $\mu(g)$ is the degree to which g satisfies Conditions (1) and (2).

In terms of g, the probability of the fuzzy event "X is $^*(a+b)/2$" may be expressed as

$$r = \int_R \mu^* a(a+b)/2(u) g(u)\, \mathrm{d}u.$$

Similarly, the expected value of X may be expressed as

$$E(X) = \int_R u g(u)\, \mathrm{d}u.$$

Using the extension principle, computation of r reduces to solution of the variational problem

$$\mu_r(v) = sup_g \ \mu(g)$$

subject to

$$v = \int_R \mu^*(a+b)/2(u) g(u)\, \mathrm{d}u$$

and

$$\int_R g(u)\, \mathrm{d}u = 1.$$

Similarly,

$$\mu_{E(X)}(v) = sup_g \ \mu(g)$$

subject to

$$v = \int_R u g(u)\, \mathrm{d}u$$

and

$$\int_R g(u)\, \mathrm{d}u = 1.$$

An important issue enters our analysis. The fuzzy set whose membership function is $\mu(g)$, may be subnormal, i.e., may have a partial existence. In this event, r and $E(X)$ should be treated as attributes of an object which exists to a degree. In the context of natural languages, a brief discussion of this nontrivial issue may be found in Zadeh [2]. Basically, in problems involving partial existence it may be necessary to move from scalar membership functions to vector-valued membership functions or, more generally to L-fuzzy sets.

So what are the central points of my comment? First, imprecise probabilities in the sense defined in de Cooman's paper are a special case of granular probabilities. Second, when we analyze a problem in which the initial probabilities are imprecise—as they are in Examples (a), (b) and (c)—the induced probabilities are, in general, granular. In other words, the class of imprecise probabilities in de Cooman's sense is not closed under imprecisely defined operations. And third, computation of granular probabilities requires concepts and techniques drawn from the theory of generalized constraints and, more particularly, from GCL and PNL.

In conclusion, Dr. de Cooman wrote a paper which is precise, rigorous and erudite. My question is: Can his elegant paper deal effectively with imprecise probabilities in real-world settings in which everything, and not just probabilities, is imprecise?

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